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The effects of nonlinearity in turning operation

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Abstract. This investigation is concerned with the effects of nonlinearity in turning operation with chatter. The turning operation is considered as a one-degree-of-freedom model and the instantaneous cutting force is expressed as a function of a linear chip thickness and nonlinear feed-rate variations. By use of the mathematical theory of delay dynamical systems, the governed neutral delay differential equations (NDDEs) for the turning model are reduced to a family of ordinary differential equations (ODEs) at a Hopf bifurcation point where steady state turning operation changes from stable to unstable. The computation of the conditions of Hopf from the linearized NDDEs yields the values necessary to establish the bifurcation point. Furthermore, by means of the integral averaging method, explicit bifurcation equations of the form $\Im(a, \mu) = 0$, where a is the chatter amplitude and μ is depth of cut, are derived. From the signs of the characteristic exponents of $\Im(a, \mu) = 0$, the regions of stable and unstable nonlinear turning operation are obtained.

Key words: nonlinear machine-tool chatter, Hopf bifurcation, centre manifold theorem, integral averaging method, stable and unstable machining operations.

1. Introduction

Chatter is an instability in machining dynamics arising often during chip-removal processes when the tool is cutting a surface $x(t)$ at time t that is already modulated from the pre-machined surface profile $x(t - h)$ at time $t - h$, in which $h := 2\pi/\Omega$ is the time delay between successive tool cuts and Ω is the spindle speed. Referring to the one-degree-of-freedom model in Figure 1, which Tobias [1] considered to represent an orthogonal turning operation, the cutting tool is removing chips from the workpiece with a nominal chip thickness s_0 as indicated by the dashed lines. The resulting surface profile $x(t)$ for this situation is flat for all values of time t and the required cutting force has a constant value. If, on the other hand, the turning process is excited by abrupt changes in the cutting force, such as those generated by a hard spot in the workpiece, or the phase shift between the modulations of successive cuts, then the surface profile of the chip to be machined from this pre-cut surface $x(t)$ is no longer flat, but has a wavy form $x(t - h)$ that is produced at time $t - h$. Consequently, sufficient energy stored in the machine-tool and the cutting process will be released to throw the turning operation into a chatter vibratory motion. From time t onwards s_0 is altered by an instantaneous value Δs of the chip thickness in order to produce the so-called regenerative chatter chip thickness $\Delta s = x(t) - x(t - h)$.

Chatter reduces productivity owing to the scenario of events such as regeneration waviness of the marks on the workpiece, increased tool wear, poor dimensional accuracy, and gradual breaking of the tool and machine-tool components, which are produced as result of its

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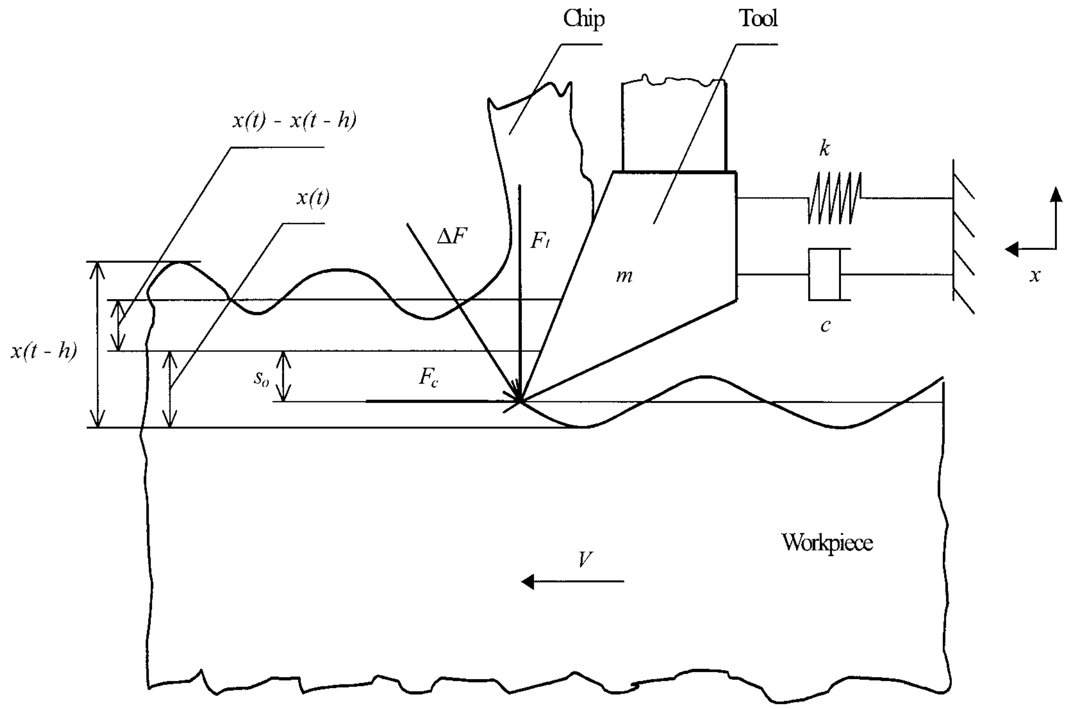

Figure 1. Orthogonal turning operation in a chatter situation.

presence in the chip removal processes. These events in turn are accompanied with fluctuating nonlinear delayed cutting forces which are considered as major routes to a variety of nonlinear chatter phenomena, ranging from finite-amplitude instability [2–5] to random chatter [6–9]. Attempts to understand these phenomena have led to a number of experimental and theoretical investigations, and the corresponding equations of motion are typically governed by delay differential equations of the retarded and/or the neutral types. They are called retarded delay differential equations (RDDEs), if the delay differential equations possess a time-delay action in the restoring force, while those differential equations with a time-delay action in the damping force are called neutral delay differential equations. Both of these equations differ from the traditional ODEs, in that solutions to ODEs are usually independent of a system's past history, while solutions to delay differential equations require the specification of an initial continuously differentiable function in the closed interval $[-h, 0]$. Solutions to DDEs are difficult to derive and moreover, they are known to exhibit complex dynamics. For example, a one-dimensional scalar linear ODE $\dot{x}(t) = -\mu x(t)$ has the algebraic characteristic equation $\Delta_{ODE} := \lambda + \mu = 0$, while the same ODE with the time delay h namely $\dot{x}(t) = -\mu x(t-h)$, has the transcendental characteristic equation $\Delta_{DDE} := \lambda + \mu e^{-h\lambda} = 0$. It can be readily seen that $\Delta_{\mathrm{ODE}} = 0$ has one eigenvalue and $\Delta_{\mathrm{DDE}} = 0$ has an infinite number of eigenvalues for a fixed value of $\mu \neq 0$. In this regard, there are inherent qualitative differences between the dynamics of the two scalar differential equations $\dot{x}(t) = -\mu x(t)$ and $\dot{x}(t) = -\mu x(t - h)$. Thus the modelling of a physical system with time-delay actions by ODEs will no doubt lead to a significant loss of dynamics.

In their efforts to understand finite-amplitude instability, Hanna and Tobias [2] conducted experimental tests for orthogonal turning operation and formulated the operation as a single-degree-of-freedom-model. They derived the nonlinear retarded delay equation $m\ddot{x} + c\dot{x} + \Gamma(x) = -\Delta f$, consisting of mass m, linear damping c, nonlinear spring force $\Gamma(x) = k_1 x + k_2 x^2 + k_3 x^3$, and the instantaneous cutting force as $\Delta f = \sum_{j=1}^{p} \sigma_j \{\Delta s\}^j$, where $\Delta s = x(t) - x(t-h)$ for $\Delta s > -s_0$ and $p = 3$. Δs is the variation of the chip thickness s from its nominal value s_0. k_1, k_2, k_3, σ_j are the model coefficients. The authors showed that when $\Delta s \leq -s_0$ the cutting tool leaves the workpiece material over a part of the chatter cycle where the response amplitudes exceed certain nominal values. While for $\Delta s > -s_0$ the authors observed that the cutting process commenced to chatter with rapidly growing amplitudes which ultimately stabilized at some finite level. This stabilization is called finite-amplitude instability. From their investigations it was clear that this instability occurs primarily as a consequence of the nonlinearity of the cutting force being a function of the chip thickness and/or the feed rate. In this regard, Shi and Tobias [3] used only the linear spring force, and the nonlinear cutting force due to the chip thickness and feed rate variations to examine finite-amplitude instability in milling. The regeneration chatter marks on the machined surface profile and the associated dynamics were found to vary also in accordance with the nonlinearities in the cutting force. Using this same equation of Shi and Tobias without the feed-rate effect, Stépán and Nagy [5] demonstrated theoretically the existence of periodic orbits leading to super- and subcritical bifurcations during orthogonal turning operation. As the chip thickness was varied, they found that chatter machining was globally stable for particular cutting conditions as long as the depth of cut remained 8% below its nominal value at Hopf bifurcation. Their results, which were captured in an infinite-dimensional Banach space, have established the notion that instability chatter machining is consistent with the typical bifurcational changes (stable and unstable limit cycles, super- and subcritical bifurcations) appearing in nonlinear dynamical systems.

In this paper we restrict ourselves to the investigation of a one-degree-of-freedom orthogonal turning model when the turning operation loses its stability and undergoes super- and subcritical bifurcations to exhibit finite-amplitude instability. A variant form of the neutral delay differential equations formulated for orthogonal turning operation by Nigm *et al.* [10], who did not take into account the nonlinear effects, will be utilized. For simplicity we will consider only the chatter vibration arising in the direction of the main cutting force. Following the inspiring work in [5], first, we will identify the onset of chatter (a Hopf bifurcation point as it will be called, henceforth, in this paper) by studying the transcendental characteristic equation associated with the linearized turning model when the depth of cut is continuously varied. Then the super- and subcritical bifurcations occurring at the bifurcation point are analyzed by the construction of a two-dimensional centre manifold in the infinite-dimensional Banach space $C([-h, 0], \mathcal{R}^2)$ for a fixed and positive time delay $h > 0$.

2. Constructing the centre manifold

The modelling of chatter in chip-removal processes often leads to delay differential equations of the retarded type, whose stability behaviour can be adequately defined in a generalized centre manifold. To construct such a manifold, we employ the axiomatic standard notations of Hale [12] and represent the resulting tool displacements due to chatter experienced at times $t - h$ and t in a particular directional machining as a functional retarded differential equation of the form

$$\dot{x}(t) = L(x_t(\theta), \mu)x_t(\theta) + \varepsilon \triangle f(x_t(\theta), \mu, \varepsilon), \quad t \geq 0, \quad 0 \leq \varepsilon \ll 1, \tag{2.1}$$

with the definition $x_t(\theta) = x(t+\theta)$, $-h \le \theta \le 0$. The element $x_t(\theta) \subseteq C := C([-h, 0], \Re^n)$ is the past history of the future state variable $x(t)$ in $\Re^n$. $L = L(x_t(\theta), \mu) : C \times \Re \to \Re^n$ is a linear functional mapping and for a prescribed function $\eta(\theta, \mu) : [-h, 0] \to \Re^n$ with elements of bounded variation in $[-h, 0]$, specifically, $\eta(\theta, \mu) = \{-L((-h), \mu)$, when $\theta = -h$, 0, when $-h < \theta < 0$ and $L((0), \mu)$ when $\theta = 0$, L may be expressed in integral form as $L(x(\theta), \mu)x_t(\theta) = \int_{-h}^{0} [d\eta(\theta, \mu)]x_t(\theta)$. Here μ is the bifurcation parameter that represents the depth of cut. The perturbation $\triangle f = \triangle f(x_t(\theta), \mu, \varepsilon) : \Re \times C \to \times^n$ is a strictly nonlinear functional mapping and it is continuous with respect to its arguments.

As in linear ODEs with constant coefficients, the linear delay equation $\dot{x}(t) = L(x_t(\theta), \mu)$ of Equation (2.1) has the transcendental characteristic equation $\Delta(\lambda, \mu) := \det\{\lambda I - \int_{-h}^{0} [d\eta(\theta, \mu)]e^{\lambda\theta}\} = 0$, where I is the identity matrix. The eigenvalues of $\Delta(\lambda, \mu) = 0$ are infinite and they may be real, or, occur in complex conjugate pairs. We will assume that a finite number of eigenvalues, say $\wedge(\lambda, \mu) = \{\lambda_1, \lambda_2, \dots \lambda_k\}$, $k = 1, 2, 3, \dots n$, having zero real part, are the eigenvalues of $\Delta(\lambda, \mu) = 0$ and all other eigenvalues have negative real parts. For such characterization, Hale and his colleagues [11–16] have shown that there exits the direct sum decomposition of C into two disjoint subspaces by all the eigenvalues of $\Delta(\lambda, \mu) = 0$ as $C = P(\lambda, \mu) \oplus Q(\lambda, \mu)$. Here $P = P(\lambda, \mu)$ is the generalized eigenspace spanned by the linear solutions corresponding to the finite number of eigenvalues $\wedge(\lambda, \mu) = \{\lambda_1, \lambda_2, \dots \lambda_k\}$, while $Q = Q(\lambda, \mu)$ is the infinite-dimensional, complementary subspace associated with the remaining eigenvalues of $\Delta(\lambda, \mu) = 0$. In addition, they showed that there exits a k-dimensional local centre manifold $M_\mu = M_\mu(\lambda, A(\theta, \mu))$ in C, whose linear extension is the generalized eigenspace P. P and M_μ are equivalent, in that, they are both the generalized eigenspace in C, and moreover, the solution of (2.1) in C through a prescribed initial continuously differentiable function, say $\phi(\theta) \subseteq C$ with initial value $\phi(0)$ at zero, projected onto P, is the corresponding solution in the centre manifold M_μ.

Indeed, we can say that $x(\phi(0), t, \mu, \varepsilon)$ is a solution of (2.1) through the function $\phi(\theta)$ if and only if, for all values of $t \in (-\infty, \infty)$, $x_t(\phi(\theta), \mu)$ satisfies the variation of constants-integral equation $x_t(\phi(\theta), \mu, \varepsilon) = J(t, \mu)\phi(\theta) + \varepsilon \int_{0}^{t} J((t-\zeta), \mu)X_0(\theta)\triangle f(x_\zeta(\theta), \mu, \varepsilon)d\zeta$, where $J(t, \mu)\phi(\theta)$ is the solution operator of the linear delay equation $\dot{x}(t) = L(x_t(\theta), \mu)x_t(\theta)$ and the integral part denotes the solution operator of the nonlinear part of (2.1). $J(t, \mu)\phi(\theta)$ is a solution operator because, for $\phi(\theta) \subseteq C$, we have the mapping of the past history $x_t(\theta)$ into the future by the relation $x_t(\phi(\theta)) = J(t, \mu)\phi(\theta)$, where the function $\phi(\theta)$ is the linear combination of the solutions corresponding to the finite eigenvalues of $\Delta(\lambda, \mu) = 0$. $J(t, \mu)$, $t, \mu \ge 0$ is a strongly continuous semigroup of bounded linear operators in C with infinitesimal generator $A(\theta, \mu)$ defined accordingly as $D(A(\theta, \mu)) = \{\phi(\theta) \subseteq C, \dot{\phi}(0) = L(\phi(\theta), \mu)\phi(\theta)\}$ and $A(\theta, \mu)\phi(\theta) = \dot{\phi}(\theta)$. The semigroup properties of $J(t, \mu)$ imply that the spectra sets $\sigma(A(\theta, \mu))$, of $A(\theta, \mu)$ and $\sigma(J(t, \mu))$ of $J(t, \mu)$ are the point spectra of finite type, and have eigenvalues with zero real parts satisfying the transcendental characteristic equation $\Delta(\lambda, \mu) = 0$. The element $X_0(\theta)$ in the integral equation is defined as $X_0(\theta) = 0$, $-h \le \theta < 0$, $X_0(0) = I$, $\theta = 0$. P and Q are both invariant under $J(t, \mu)$ and $A(\theta, \mu)$, which means that a solution in C starting from a point in either P or Q, will indeed always remain in P for all $t \in (-\infty, \infty)$, or, Q for all $t \in [0, \infty)$. In particular, with the integral solution $x_t(\phi(\theta), \mu, \varepsilon)$ of (2.1) in C, we have the unique representation

$x_t(\phi(\theta),\mu,\varepsilon) = x_t^P(\phi(\theta),\mu,\varepsilon) + x_t^Q(\phi(\theta),\mu,\varepsilon)$, where $x_t^P(\phi(\theta),\mu,\varepsilon)$, $x_t^Q(\phi(\theta),\mu,\varepsilon)$ are the projections of $x_t(\phi(\theta),\mu,\varepsilon) \subseteq C$ onto the two subspaces P, Q, respectively. Similarly, for $\phi(\theta)$, $X_0(\theta) \subseteq C$, we have the representation $\phi(\theta) = \phi^P(\theta) + \phi^Q(\theta)$, where $\phi^P(\theta) = \Phi(\theta)b$, and $X_0(\theta) = X_0^P(\theta) + X_0^Q(\theta)$, where $X_0^P(\theta) = \Phi(\theta)\Psi(s)$ and b is some constant vector. The function $\Phi(\theta) \subseteq C$ is a basis to all the solutions of $\dot{x}(t) = L(x_t(\theta),\mu)x_t(\theta)$ in M_μ for which $\Delta(\lambda,\mu) = 0$ has the k eigenvalues with zero real parts, whereas $\Psi(s) \subseteq \hat{C}$, $0 \le s \le h$ is the basis for all solutions of the linear functional delay differential equation $\dot{u}(\hat{t}) = \hat{L}(u_{\hat{t}}(s),\mu)u_{\hat{t}}(s)$, $u(\hat{t}) \in \Re^n$ in the Banach space $\hat{C} := \hat{C}([0,h],\Re^n)$. This equation in $\hat{C}$ is formally adjoint to $\dot{x}(t) = L(x_t(\theta),\mu)x_t(\theta)$ with respect to the bilinear relation $(\psi_j(s),\phi_k(\theta)) = (\psi_j(0),\phi_k(0)) - \int_{-h}^{0}\int_0^\theta \psi_j(\zeta - s)[\mathrm{d}\eta(\theta,\mu)]\phi_k(\zeta)\mathrm{d}\zeta$, j, $k = 1,2,\ldots n$, for $\phi_k(\theta) \subseteq C$ and $\psi_j(s) \subseteq \hat{C}$. The linear functional $\hat{L} = \hat{L}(u_{\hat{t}}(s),\mu) : \hat{C} \times \Re^n \to \Re^n$ is the transpose of $L(x_t(\theta),\mu)$ and for the bounded variational function $\hat{\eta}(s,\mu) : [0,h] \to \Re^n$, $\hat{L}$ is described by $\hat{L}(u_{\hat{t}}(s),\mu) = -\int_{-h}^{0}[\mathrm{d}\hat{\eta}(s,\mu)]u_{\hat{t}}(s)$. Although a unique solution for the adjoint equation can be obtained by specifying an initial continuously differentiable function $\psi(s)$ in $\hat{C}$, it is interesting to note that the corresponding transcendental characteristic equation $\hat{\Delta}(\lambda,\mu) = 0$ has exactly the same eigenvalues as those of $\Delta(\lambda,\mu) = 0$ in C. The corresponding semigroup $J(\hat{t},\mu)$, $\hat{t}$, $\mu \ge 0$, and its infinitesimal generator $\hat{A}(s,\mu)$, defined by $\hat{D}(\hat{A}(s,\mu)) = \{\psi(s) \subseteq \hat{C},\ \dot{\psi}(0) = \hat{L}(\psi(s),\mu)\psi(s)\}$ and $\hat{A}(s,\mu)\psi(s) = \psi(s)$, of $\dot{u}(\hat{t}) = \hat{L}(u_{\hat{t}}(s),\mu)u_{\hat{t}}(s)$ have also the point spectra sets of the finite type with eigenvalues satisfying $\Delta(\lambda,\mu) = 0$. In particular, for the finite eigenvalues $\wedge(\lambda,\mu) = \{\lambda_1,\lambda_2,\ldots\lambda_k\}$ with zero real parts, there is also the generalized eigenspace $\hat{P} = \hat{P}(\lambda,\mu)$ and the local centre manifold $\hat{M}_\mu = \hat{M}_\mu(\lambda,\hat{A}(s,\mu))$ in $\hat{C}$. Moreover for $\psi(s)$ in $\hat{C}$, we have its projection onto $\hat{P}$ as $\psi^{\hat{P}}(s) = \Psi(s)\hat{b}$, where $\hat{b}$ is some constant vector.

Incidently, because of the fact that the two linear equations $\dot{x}(t) = L(x_t(\theta),\mu)x_t(\theta) \subseteq C$ and $\dot{u}(\hat{t}) = \hat{L}(u_{\hat{t}}(s),\mu)u_{\hat{t}}(s) \subseteq \hat{C}$ are adjoint, we have the identity $(\Psi(s), A\Phi(\theta)) \equiv (\hat{A}\Psi(s),\Phi(\theta))$, where for some $k \times k$ matrices $B \subseteq C$, $\hat{B} \subseteq \hat{C}$ whose real elements are the eigenvalues of $\wedge(\lambda,\mu)$, and are obtained from the fact that $A\Phi(\theta) = \Phi(\theta)B$, $\hat{A}\Psi(s) = \Psi(s)\hat{B}$; we have $(\Psi(s),A\Phi(\theta)) \equiv (\Psi(s),\Phi(\theta)B) \equiv (\Psi(s),\Phi(\theta))B$, $(\hat{A}\Psi(s),\Phi(\theta)) \equiv (\Psi(s)\hat{B},\Phi(\theta)) \equiv \hat{B}(\Psi(s),\Phi(\theta))$. Thus the two real matrices B, $\hat{B}$ are equivalent provided the inner product $(\Psi(s),\Phi(\theta))$, whose elements are the bilinear relation $(\psi_j(s),\phi_k(\theta))$, is the identity matrix. However, $(\Psi(s),\Phi(\theta))$ is generally not equal to the identity matrix, that is $(\Psi(s),\Phi(\theta)) \ne I$. In such a situation, the matrices B, $\hat{B}$ are related as $B = (\Psi(s),\Phi(\theta))^{-1}$ $\hat{B}(\Psi(s),\Phi(\theta))$, and thus the basis $\Psi(s)$ for $\hat{P} \subseteq \hat{C}$ can be readily normalized to a new set of functions $\bar{\Psi}(s) \subseteq \hat{C}$ by carrying out the algebraic computation of $\bar{\Psi}(s) = (\Psi(s),\Phi(\theta))^{-1}\Psi(s)$. This new basis $\bar{\Psi}(s)$ will indeed ensure that the inner product $(\bar{\Psi}(s),\Phi(\theta)) = I$.

With the space C decomposed as above along with $\Phi(\theta)$, $\Psi(s)$ being bases for $P \subseteq C$, $\hat{P} \subseteq \hat{C}$, and in addition, we have that $\Psi(s)$ in $\hat{C}$ is normalized to $\bar{\Psi}(s)$ such that $(\bar{\Psi}(s),\Phi(\theta)) = I$. Then the elements needed to write the integral equation $x_t(\phi(\theta),\mu)$ as a solution in the centre manifold in C are given by $P = \{(\phi(\theta) \in C,\ \phi(\theta) = \phi^P(\theta) + \phi^Q(\theta) \mid \phi^P(\theta) = \Phi(\theta)b,\ b := (\bar{\Psi}(s),\phi^P(\theta))\}$, while those elements in the complementary space are given by $Q = \{(\phi(\theta) \in C,\ \phi^Q(\theta) = \phi(\theta) - \phi^P(\theta) \mid (\bar{\Psi}(s),\phi^Q(\theta)) = 0\}$. Therefore, we have an equivalent solution of (2.1) in M_μ as $x_t(\phi(\theta),\mu,\varepsilon) = \Phi(\theta)y(t) + x_t^Q(\phi(\theta),\mu,\varepsilon)$, where $y(t) \in \Re^n$ defined by $y(t) = (\bar{\Psi}(s),\phi(\theta))$ satisfies a family of ODEs. In order to obtain such ODEs, we take up the derivative of $x_t(\phi(\theta),\mu,\varepsilon) = \Phi(\theta)y(t) + x_t^Q(\phi(\theta),\mu)$ and impose it onto the variation-of-constant integral equation $x_t(\phi(\theta), \mu, \varepsilon) = J(t, \mu)\phi(\theta) + \varepsilon \int_0^t J((t - \zeta), \mu)X_0(\theta)\triangle f(x_\zeta(\theta), \mu, \varepsilon)\mathrm{d}\zeta$ in C. This will yield the k-dimensional ODEs in $M_\mu \subseteq C$ as follows:

$$\dot{y}(t) = By(t) + \varepsilon\bar{\Psi}(0)\triangle f(\Phi(\theta)y(t), \mu, \varepsilon), \quad -h \le t < \infty, \tag{2.2a}$$

and the integral solution in the infinite-dimensional subspace $Q \subseteq C$ is

$$\begin{aligned} x_t^Q(\phi(\theta), \mu, \varepsilon) = {} & J(t, \mu)\{\phi(\theta) - \phi^P(\theta)\} + \varepsilon \int_0^t J((t - \zeta), \mu)\{X_0(\theta) - \\ & X_0^P(\theta)\}\triangle f(x_\zeta(\theta), \mu, \varepsilon)\mathrm{d}\zeta, \quad 0 \le t < \infty, \end{aligned} \tag{2.2b}$$

where B is the $k \times k$ real matrix. As a first-order approximation, Hale and his colleagues [11–16] have suggested that long-term behaviour of Equation (2.1) is, in principle, well approximated by the k-dimensional ODEs (2.2a) corresponding to the integral equation $x_t^P(\phi(\theta), \mu, \varepsilon)$ of $P \subseteq C$ in the centre manifold M_μ. There are two seminal reasons for such an assertive argument. First, the authors stated that the integral solution $x_t(\phi(\theta), \mu, \varepsilon)$ in Q, in general, cannot satisfy Equation (2.1), because $x_t^Q(\phi(\theta), \mu, \varepsilon)$ does not satisfy the fundamental property $x_t^Q(\theta) \ne x_t^Q(t + \theta)$, $-h \le \theta \le 0$ of functional differential equations. Secondly, they have demonstrated that, as the bifurcation parameter μ varies in the vicinity of its critical value μ_c, and moreover, as $t \to \infty$, the exponential estimate of the integral solution in Q, namely $\mid x_t^Q(\phi(\theta), \mu, \varepsilon) \mid = \mid x_t(\phi(\theta), \mu, \varepsilon) - x_t^P(\phi(\theta), \mu, \varepsilon) \mid \to 0$. Therefore, there is no loss in generality by assuming that the dynamics of practical interest occurs in the smooth k-dimensional centre manifold M_μ. In a similar way, the investigations of the stability and bifurcations of equilibria in a multiple-delay differential equations with nonsymmetric nonlinearities by Bélair and Sue Ann Campbell [17], and the nonlinear regenerative chatter with cubic, symmetric nonlinearities by Stépán and Nagy [5] showed that the corresponding ODEs in the prescribed local centre manifolds were sufficient to understand the dynamics of the physical systems of interest. We feel that the trade-off of the integral solution $x_t^Q(\phi(\theta), \mu, \varepsilon)$ in Q is far better than the many previous techniques used to reduce delay equations to finite ODEs under the naive assumption of small or negligible time delay. This theory of retarded delay differential equations, which has also been extended to NDDEs by Hale [12–14], is a consequence of extensive general results for the study of delay dynamical systems in an infinite-dimensional space. The recent book [13] and the references cited therein will provide a detailed account of the theories for both the retarded and neutral DDEs.

In the sequel, we will explore the bifurcation of a single degree of freedom turning model leading to finite-amplitude stability in the infinite-dimensional space $C := C([-h, 0], \Re^2)$ when the finite eigenvalues $\wedge(\lambda, \mu) = \{\lambda_1, \lambda_2, \ldots \lambda_k\}$ of $\Delta(\lambda, \mu) = 0$ has the two complex conjugates eigenvalues, namely $\wedge(\lambda, \mu) = \upsilon(\mu) \pm i\omega(\mu)$. Furthermore, we will assume that these eigenvalues satisfy the classical conditions of Hopf [12, 13]. That is, for $\mu = \mu_c$, we have $\upsilon(\mu_c) = 0$, $\omega(\mu) \ne 0$, and by the implicit-function theorem we have the crossing velocity $\mathfrak{Re}\{\mathrm{d}\Delta(\lambda, \mu)/\mathrm{d}\mu\}_{\mu=\mu_c} \ne 0$ of the eigenvalues from left to right in the complex plane.

3. Bifurcations and finite amplitude instability

In this section, we examine super- and subcritical bifurcations leading to finite-amplitude instability by adopting nonlinear neutral delay differential equations, which are somehow similar to those equations formulated by Tobias and his colleagues [2, 3, 10]. The cutting force is considered here as a function of a linear chip thickness and nonlinear feed rate. In this regard, we have the nondimensional functional delay differential equation

$$\begin{aligned}&\Omega^2\ddot{x} + \delta_0\Omega\dot{x} + \beta_1 x + \mu\{\beta_2(x - x(t-h)) + \\ &\quad \delta_1(\dot{x} - \dot{x}(t-h))\} = \varepsilon\sum_{j=2}^{q}\alpha_j\{\dot{x} - \dot{x}(t-h)\}^j,\end{aligned} \tag{3.1}$$

in $C := C([-h, 0], \Re^2)$, where Ω is the spindle speed, the depth of cut μ is subjected to small perturbation $\varepsilon\tilde{\mu}$ from its critical value μ_c and $\delta_0,\ \delta_1,\ \beta_1,\ \beta_2$ are the physical parameters of the turning model. The coefficients $\alpha_j,\ j = 2,\ 3,\ 4, \ldots q$ characterize the nonlinear nature of the cutting force $\sum_{j=2}^{q}\alpha_j\{\dot{x} - \dot{x}(t-h)\}^j$. Nam *et al.* [10] have utilized the left-hand side of this equation, in order to illustrate the effects of the linear variations of the feed rate on orthogonal turning operation. Shi and Tobias [3] showed that, in the absence of the delayed damping term $\dot{x}(t-h)$, the regions of linear stable milling operations increased as the feed rate factor δ_1 was continuously varied. Recently, Baker and Rouch [18] have also expressed the cutting force for orthogonal turning process in the same form as the left-hand side of (3.1). Using a finite-element method and varying the flexibility of both the tool and the workpiece, they were able to identify stability regions and the operating frequencies within such regions for a small range of spindle speeds. Efforts to explore the nonlinear variation of the feed rate are very limited and the precise nature of the degree of nonlinearity of the cutting force as a function of the feed rate is so far not known in the literature. It is known, however, for $j = 2, 3$ that the nonlinear cutting force in Equation (3.1) is similar to the well-known nonlinear cutting force as a function of the chip thickness, which has been formulated by Tobias and his colleagues [2, 3] for turning and milling operations. Therefore, for the sake of exploring the bifurcations leading to finite-amplitude instability, we will assume that the degree of nonlinearity of the cutting force is of the nonsymmetric form, namely, $j = 3$ and $j = 5$ in Equation (3.1).

The linear part of Equation (3.1)

$$\Omega^2\ddot{x} + \delta_0\Omega\dot{x} + \beta_1 x + \mu\{\beta_2(x - x(t-h)) + \delta_1(\dot{x} - \dot{x}(t-h))\} = 0, \tag{3.2a}$$

in C has the characteristic equation

$$\Delta(\lambda, \mu) := \Omega^2\lambda^2 + (\delta_0\Omega + \delta_1\mu)\lambda + (\beta_1 + \beta_2\mu) - (\beta_2 + \delta_1\lambda)\mu e^{-\lambda h} = 0. \tag{3.2b}$$

The eigenvalues of this equation are the same as those corresponding to the adjoint equation

$$\Omega^2\ddot{x} - \delta_0\dot{x} + \beta_1 x + \mu\{\beta_2(x - x(\hat{t} + h)) - \delta_1(\dot{x} - \dot{x}(\hat{t} + h))\} = 0, \tag{3.3a}$$

in $\hat{C} := \hat{C}([0, h], \Re^2)$ with respect to the bilinear relation

$$\begin{aligned}&(\psi_j(s), \phi_k(\theta)) = \psi_j(0)\phi_k(0) - \frac{\delta_1\mu}{\Omega^2}\{\psi_j(0)\phi_k(-h)\} - \\ &\frac{\delta_1\mu}{\Omega^2}\int_{-h}^{0}\left(\frac{d\psi_j(h+\zeta)}{d\zeta}\right)\phi_k(\zeta)d\zeta + \frac{\beta_2\mu}{\Omega^2}\int_{-h}^{0}\psi_j(h+\zeta)\phi_k(\zeta)d\zeta, \quad j, k = 1, 2,\end{aligned} \tag{3.3b}$$

for $\phi_k(\theta) \subseteq C$ and $\psi_j(s) \subseteq \hat{C}$. Substituting $\Lambda(\lambda, \mu)\ |_{\mu=\mu_c} = \pm \mathrm{i}\omega$ in (3.2b), and simultaneously solving the real and imaginary equations for μ, Ω will give rise to the following conditions for the onset of chatter at the Hopf bifurcation point $\mu = \mu_c$:

$$\begin{aligned} \mu &= -\frac{\delta_0 \omega \Omega}{\delta_1 \omega (1 - \cos \omega h) + \beta_2 \sin \omega h}, \\ \omega^2 \Omega^2 &- \frac{\delta_0 \beta_2 (1 - \cos \omega h) - \delta_0 \delta_1 \omega \sin \omega h}{\delta_1 \omega (1 - \cos \omega h) + \beta_2 \sin \omega h} \omega \Omega + \beta_1 = 0, \end{aligned} \tag{3.4a}$$

where by the implicit-function theorem we have the crossing condition

$$\begin{aligned} \mathfrak{Re} \left\{ \frac{\mathrm{d}\Delta(\lambda, \mu)}{\mathrm{d}\mu} \right\}_{\mu=\mu_c} = &-\frac{1}{R_{111}^2 + R_{112}^2} \{R_{111}\{\beta_2(1 - \cos \omega h) - \delta_1 \omega \sin \omega h\} + \\ &R_{112}\{\delta_1 \omega (1 - \cos \omega h) + \beta_2 \sin \omega h\}\} > 0, \end{aligned} \tag{3.4b}$$

and it is positive provided $R_{111}\{.\} + R_{112}\{.\} < 0$, in which the notations R_{111}, R_{112} denote

$$\begin{aligned} R_{111} = &\ \delta_0 \Omega + \frac{1}{\delta_1 \omega (1 - \cos \omega h) + \beta_2 \sin \omega h} \{\delta_0 \omega \Omega (\delta_1 - \beta_2 h) \cos \omega h - \\ &\delta_0 \delta_1 \omega \Omega (1 + \omega h \sin \omega h)\}, \\ R_{112} = &\ 2\omega \Omega^2 - \frac{1}{\delta_1 \omega (1 - \cos \omega h) + \beta_2 \sin \omega h} \{\delta_0 \omega \Omega (\delta_1 - \beta_2 h) \sin \omega h + \\ &\delta_0 \delta_1 \omega^2 h \Omega \cos \omega h)\}. \end{aligned} \tag{3.4c}$$

Thus, for the parameter values $\beta_1 = 230$, $\beta_2 = h = 1$, $\delta_0 = 0{\cdot}0625$, we capture conditions (3.4a) and (3.4b) qualitatively in Figure 2, and from this it can be readily seen that the turning operation is stable as long as a coordinate point of the depth of cut μ and spindle speed Ω does not fall within the unstable region in the $\Omega - \mu$ plane.

Next the bifurcations leading to finite-amplitude instability in the two-dimensional centre manifold $M_\mu \subseteq C$ are determined by computing the decomposition $C = P \oplus Q$ according to the simple eigenvalues $\Lambda(\lambda, \mu)\ |_{\mu=\mu_c} = \pm \mathrm{i}\omega$ associated with the generalized eigenspaces $P \subseteq C$, $\hat{P} \subseteq \hat{C}$, as well as all the remaining eigenvalues of $\Delta(\lambda, \mu) = 0$ corresponding to the complementary space $Q \subseteq C$. Indeed, the corresponding function $\phi^P(\theta)$ in P is $\phi^P(\theta) = \Phi(\theta) b \subseteq C$, $-h \le \theta \le 0$, where $\Phi(\theta) = [\phi_1(\theta), \phi_2(\theta)]$, $\phi_1(\theta) = [\cos \omega\theta, \sin \omega\theta]^T$, $\phi_2(\theta) = [-\sin \omega\theta, \cos \omega\theta]^T$, while $\psi(s)$ in $\hat{P}$ is $\psi(s) = \Psi(s)\hat{b} \subseteq \hat{C}$, $0 \le s \le h$, where $\Psi(s) = [\psi_1(s), \psi_2(s)]$, $\psi_1(s) = [\cos \omega s, -\sin \omega s]^T$, $\psi_2(s) = [\sin \omega s, \cos \omega s]^T$. Here T stands for transpose. Using the bilinear relation (3.3b), we can calculate the elements $(\psi_j(s), \phi_k(\theta))$, $j, k = 1, 2$ of the inner product matrix $(\Psi(s), \Phi(\theta))$. Thus we obtain the elements of the matrix $(\Psi, \Phi)_{\mathrm{nsg}}$

$$\psi_{11} \equiv \psi_{22} = 1 + \frac{\delta_0 \omega}{\Omega\{\beta_1 \omega (1 - \cos \omega h) + \beta_2 \sin \omega h\}} \{(\delta_1 - \beta_2 h) \cos \omega h - \delta_1 \omega h \sin \omega h\}, \tag{3.5a}$$

$$\begin{aligned} \psi_{12} &= \frac{\delta_0 \omega}{\Omega\{\beta_1 \omega (1 - \cos \omega h) + \beta_2 \sin \omega h\}} \{(\delta_1 - \beta_2 h) \sin \omega h + \delta_1 \omega h \cos \omega h\}, \\ \psi_{21} &= -\psi_{12}, \end{aligned} \tag{3.5b}$$

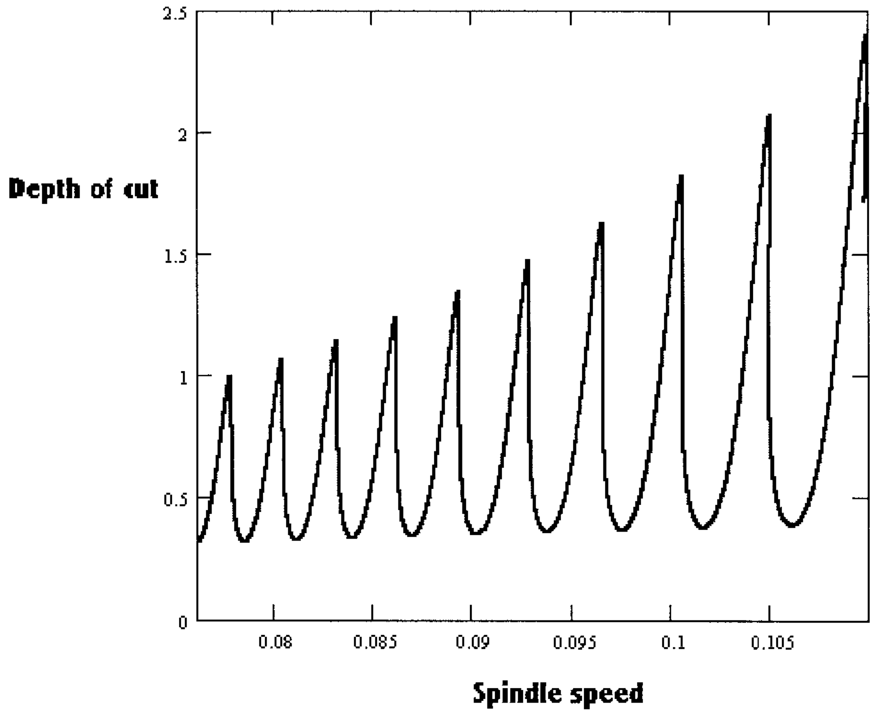

Figure 2. The linear stability chart for the turning operation.

and it is a nonsingular matrix, that is $(\Psi, \Phi)_{\text{nsg}} \neq I$. Then, after another tedious calculation of $\bar{\Psi}(s) = (\Psi, \Phi)^{-1}_{\text{nsg}}\Psi(s)$ for the normalization of $\Psi(s) \subseteq \hat{C}$ to $\bar{\Psi}(s) = [\bar{\psi}_1(s), \bar{\psi}_2(s)] \subseteq \hat{C}$, where $\bar{\psi}_1(s) = [\bar{\psi}_{11}, \bar{\psi}_{21}]^T$, $\bar{\psi}_2(s) = [\bar{\psi}_{12}, \bar{\psi}_{22}]^T$, we have

$$\begin{aligned} \bar{\psi}_{11} &= \frac{\psi_{22}\cos\omega s + \psi_{12}\sin\omega s}{\psi_{11}^2 + \psi_{12}^2}, \quad \bar{\psi}_{12} = \frac{\psi_{22}\sin\omega s - \psi_{12}\cos\omega s}{\psi_{11}^2 + \psi_{12}^2}, \\ \bar{\psi}_{21} &= -\frac{\psi_{21}\cos\omega s + \psi_{11}\sin\omega s}{\psi_{11}^2 + \psi_{12}^2}, \quad \bar{\psi}_{22} = -\frac{\psi_{21}\sin\omega s - \psi_{11}\cos\omega s}{\psi_{11}^2 + \psi_{12}^2}. \end{aligned} \tag{3.6}$$

Again substitution of the new elements $(\bar{\psi}_j(s), \phi_k(\theta))$, $j, k = 1, 2$ of $(\bar{\Psi}(s), \Phi(\theta))$ in (3.3b) and integration will give the identity matrix $(\bar{\Psi}, \Phi) = I$. To this end, we can write $x_t^P(\phi(\theta), \mu, \varepsilon) = \Phi(\theta)y(t) + x_t^Q(\phi(\theta), \mu, \varepsilon)$, $y(t) \in \Re^2$, $y(t) = (\bar{\Psi}(s), \phi(\theta))$ to obtain the equivalent ODEs on the centre manifold $M_\mu . \in C$:

$$\dot{y}_1(t) = -\omega y_2 - \varepsilon g^1(y_1, y_2), \quad \dot{y}_2(t) = \omega y_1 - \varepsilon g^2(y_1, y_2), \tag{3.7a}$$

where the above quantities represent the following

$$\begin{aligned} g^1(y_1, y_2) :=\ & F_{511}y_2^5 + F_{512}y_1y_2^4 + (F_{311} + F_{513}y_1^2)y_2^3 + (F_{312} + F_{514}y_1^2)y_1y_2^2 + \\ & (F_{313} + F_{515}y_1^2)y_1^2y_2 + (F_{314} + F_{516}y_1^2)y_1^3 + F_{111}y_2 + F_{112}y_1, \\ g^2(y_1, y_2) :=\ & G_{511}y_2^5 + G_{512}y_1y_2^4 + (G_{311} + G_{513}y_1^2)y_2^3 + (G_{312} + G_{514}y_1^2)y_1y_2^2 + \\ & (G_{313} + G_{515}y_1^2)y_1^2y_2 + (G_{314} + G_{516}y_1^2)y_1^3 + G_{111}y_2 + G_{112}y_1\}; \end{aligned} \tag{3.7b}$$

$$F_{511} = -\frac{\omega\psi_{22}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5(1-\cos\omega h)^5, \quad F_{512} = -\frac{5\omega\psi_{22}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5(1-\cos\omega h)^4\sin\omega h,$$

$$F_{513} = -\frac{10\omega\psi_{21}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5(1-\cos\omega h)^3\sin^2\omega h,$$

$$F_{514} = -\frac{10\omega\psi_{21}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5(1-\cos\omega h)^2\sin^3\omega h, \tag{3.7c}$$

$$F_{515} = -\frac{5\omega\psi_{22}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5(1-\cos\omega h)\sin^4\omega h, \quad F_{516} = -\frac{\omega\psi_{22}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5\sin^5\omega h;$$

$$F_{311} = -\frac{\omega\psi_{22}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_3(1-\cos\omega h)^3, \quad F_{312} = -\frac{3\omega\psi_{22}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_3(1-\cos\omega h)^2\sin\omega h,$$

$$F_{313} = -\frac{3\omega\psi_{22}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_3(1-\cos\omega h)\sin^2\omega h, \quad F_{314} = -\frac{\omega\psi_{22}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_3\sin^3\omega h,$$

$$F_{111} = -\frac{\tilde{\mu}}{(\psi_{11}^2+\psi_{12}^2)}\{\delta_1\omega\psi_{22}(1-\cos\omega h) - \beta_2\psi_{12}\sin\omega h\}, \tag{3.7d}$$

$$F_{112} = -\frac{\tilde{\mu}}{(\psi_{11}^2+\psi_{12}^2)}\{\delta_1\omega\psi_{22}\sin\omega h + \beta_2\psi_{12}(1-\cos\omega h)\};$$

$$G_{511} = -\frac{\omega\psi_{21}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5(1-\cos\omega h)^5, \quad G_{512} = -\frac{5\omega\psi_{21}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5(1-\cos\omega h)^4\sin\omega h,$$

$$G_{513} = -\frac{10\omega\psi_{22}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5(1-\cos\omega h)^3\sin^2\omega h,$$

$$G_{514} = -\frac{10\omega\psi_{22}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5(1-\cos\omega h)^2\sin^3\omega h, \tag{3.7e}$$

$$G_{515} = -\frac{5\omega\psi_{21}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5(1-\cos\omega h)\sin^4\omega h, \quad G_{516} = -\frac{\omega\psi_{21}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_5\sin^5\omega h;$$

$$G_{311} = -\frac{\omega\psi_{21}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_3(1-\cos\omega h)^3, \quad G_{312} = -\frac{3\omega\psi_{21}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_3(1-\cos\omega h)^2\sin\omega h,$$

$$G_{313} = -\frac{3\omega\psi_{21}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_3(1-\cos\omega h)\sin^2\omega h, \quad G_{314} = -\frac{\omega\psi_{21}}{(\psi_{11}^2+\psi_{12}^2)}\alpha_3\sin^3\omega h,$$

$$G_{111} = -\frac{\tilde{\mu}}{(\psi_{11}^2+\psi_{12}^2)}\{\delta_1\omega\psi_{21}(1-\cos\omega h) + \beta_2\psi_{11}\sin\omega h\}, \tag{3.7f}$$

$$G_{112} = -\frac{\tilde{\mu}}{(\psi_{11}^2+\psi_{12}^2)}\{\delta_1\omega\psi_{21}\sin\omega h - \beta_2\psi_{11}(1-\cos\omega h)\}.$$

Since the centre manifold $M_\mu \subseteq C$ is tangent to the plane y_1, y_2 at the origin, and moreover the nonlinearities considered in Equation (3.1) are of the fifth and cubic orders, we use the Taylor expansion of $\hbar(\Phi(\theta)y(t)) = p_{511}[\Phi(\theta)y(t)]^5 + 5p_{512}[\Phi(\theta)y(t)]^4 + \cdots p_{515}[\Phi(\theta)y(t)]$ to further simplify (3.7) by a near identity transformation, namely

$$\begin{aligned}\dot{y}_1(t) &= -\omega y_2 + \varepsilon a\{\Gamma^1_{115}(y_1, y_2, \mu)a^4 + \Gamma^1_{113}(y_1, y_2, \mu)a^3 + \tilde{\mu}\Gamma^1_{111}(y_1, y_2, \mu)\},\\ \dot{y}_2(t) &= \omega y_1 + \varepsilon a\{\Gamma^2_{115}(y_1, y_2, \mu)a^4 + \Gamma^2_{113}(y_1, y_2, \mu)a^3 + \tilde{\mu}\Gamma^2_{111}(y_1, y_2, \mu)\},\end{aligned} \tag{3.8}$$

where $p_{511}, p_{512}, \ldots p_{515}$ are constant coefficients. The resulting coefficients $\Gamma^k_{115}(y_1, y_2, \mu)$, $\Gamma^k_{113}(y_1, y_2, \mu)$, $\Gamma^k_{111}(y_1, y_2, \mu)$ are composed of all the elements in (3.7), as well as those coefficients generated by the substitution of the Taylor expansion of $\hbar(\Phi(\theta)y(t))$. At the expense of the transformation $y_1 = a \sin\Theta,\ y_2 = -a\cos\Theta,\ \Theta = \omega t + \varphi$, we write (3.8) in terms of amplitude and phase relations, namely

$$\begin{aligned}\dot{a}(t)\sin\Theta + a\dot{\varphi}\cos\Theta = {}& \varepsilon a\{\Gamma^1_{115}(a\sin\Theta, -a\cos\Theta)a^4 + \Gamma^1_{113}(a\sin\Theta, -a\cos\Theta)a^3 +\\ & \tilde{\mu}\Gamma^1_{111}(a\sin\Theta, -a\cos\Theta)\},\\ -\dot{a}(t)\cos\Theta + a\dot{\varphi}\sin\Theta = {}& \varepsilon a\{\Gamma^2_{115}(a\sin\Theta, -a\cos\Theta)a^4 + \Gamma^2_{113}(a\sin\Theta, -a\cos\Theta)a^3 +\\ & \tilde{\mu}\Gamma^2_{111}(a\sin\Theta, -a\cos\Theta)\}.\end{aligned} \tag{3.9}$$

Solving these equations for a and φ we obtain equations for the amplitude and phase, denoted by $\dot{a}(t) = \varepsilon\Gamma_a(a\sin\Theta, -a\cos\Theta, \mu, \varepsilon)$, $\dot{\varphi}(t) = \varepsilon\Gamma_\varphi(a\sin\Theta, -a\cos\Theta, \mu, \varepsilon)$. Then, the use of the averaging operators defined according to Bogoliubov-Mitropolsky [19]:

$$\begin{aligned}\dot{a} &= \underset{t}{M}\{\Gamma_a(a\sin\Theta, -a\cos\Theta, \mu, \varepsilon)\} = \lim_{T_0\longrightarrow\infty} T_0^{-1}\int_0^{T_0} \Gamma_a(a\sin\Theta, -a\cos\Theta, \mu, \varepsilon)\mathrm{d}t,\\ \dot{\varphi} &= \underset{t}{M}\{\Gamma_\varphi(a\sin\Theta, -a\cos\Theta, \mu, \varepsilon)\} = \lim_{T_0\longrightarrow\infty} T_0^{-1}\int_0^{T_0} \Gamma_\varphi(a\sin\Theta, -a\cos\Theta, \mu, \varepsilon)\mathrm{d}t,\end{aligned} \tag{3.10a}$$

will yield the uncoupled averaged equations:

$$\dot{a} = \varepsilon a\{\Pi^1_{115}a^4 + \Pi^1_{113}a^2 + \Pi^1_{111}\tilde{\mu}\}, \quad \dot{\varphi} = \varepsilon\{\Pi^2_{115}a^4 + \Pi^2_{113}a^2 + \Pi^2_{111}\tilde{\mu}\}, \tag{3.10}$$

where the notations Π^k_{115}, Π^k_{113}, Π^k_{111} are the respective coefficients after averaging over the time interval $T_0 \in [0, 2\pi]$. The bifurcations occurring and finite-amplitude instability can now be examined.

We begin by setting $a = a_0 + \rho$, where ρ is the perturbation from the steady-state value which is obtained by solving the equations $\dot{a} = 0$ and $\dot{\varphi} = 0$. Hence, we obtain the linear and nonlinear variational equations as follows:

$$\dot{\rho}(t) = -\varepsilon\tilde{\mu}\Pi^1_{111}\rho, \quad a_0 = 0, \quad \dot{\rho}(t) = -\varepsilon\{5\Pi^1_{115}a_0^4 + 3\Pi^1_{113}a_0^2 + \Pi^1_{111}\tilde{\mu}\}\rho, \quad a_0 \neq 0. \tag{3.11a}$$

Solution of these equations will yield the characteristic exponents, or equivalently the bifurcation equations

$$\begin{aligned}\Im(a, \mu)_{a=a_0=0} &:= -\varepsilon\tilde{\mu}\{\Pi^1_{111}\}_{a=a_0=0} = 0,\\ \Im(a, \mu)_{a=a_0\neq 0} &:= -\varepsilon\{\Pi^1_{115}a_0^4 + 3\Pi^1_{113}a_0^2 + \Pi^1_{111}\tilde{\mu}\}_{a=a_0\neq 0} = 0.\end{aligned} \tag{3.11b}$$

A qualitative sketch of these bifurcation equations showing the regions of stable and unstable machining operations, is displayed in Figure 3 when the depth of cut μ varies near its critical value μ_c. In these figures, the critical depth of cut μ_c corresponds to the value when the

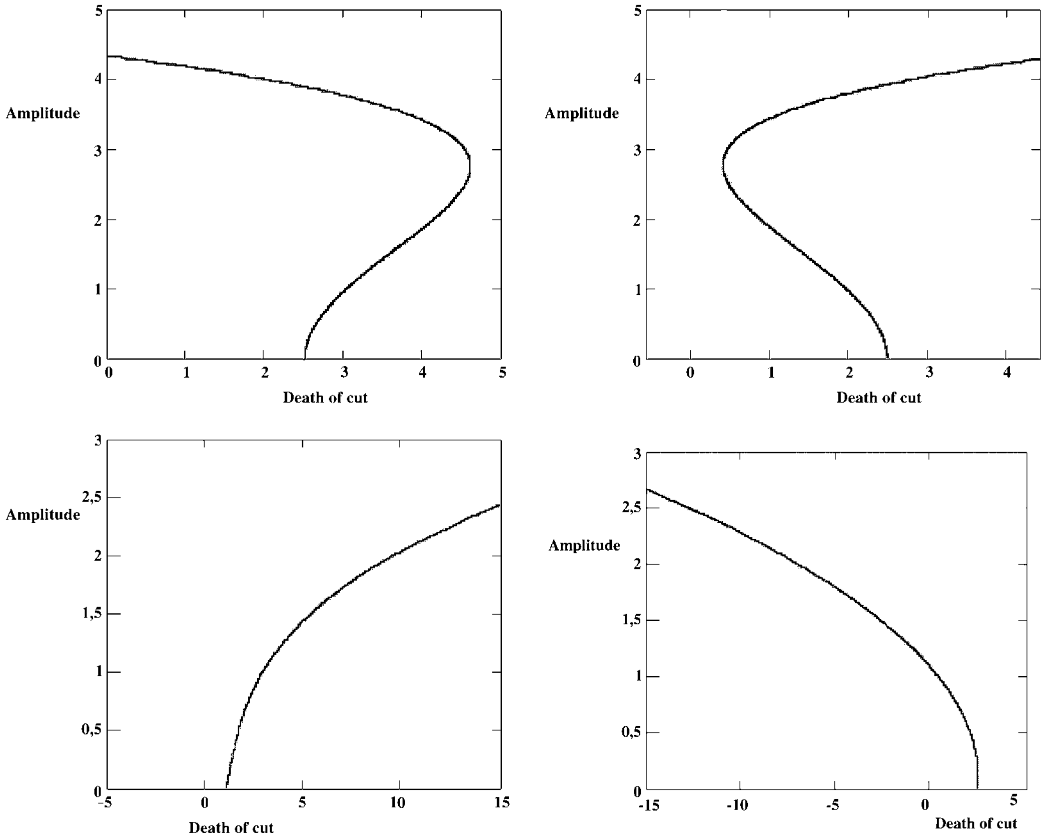

Figure 3. (a) Supercritical bifurcation for $\Pi^1_{115} < 0$ and $\Pi^1_{113} > 0$. (b) Subcritical bifurcation for $\Pi^1_{115} > 0$ and $\Pi^1_{113} < 0$. (c) Supercritical bifurcation for $\Pi^1_{115} = 0$ and $\Pi^1_{113} > 0$. (d) Subcritical bifurcation for $\Pi^1_{115} = 0$ and $\Pi^1_{113} < 0$.

amplitude $a_0 = 0$. Steady-state turning operation is stable as long as the depth of cut $\mu < \mu_c$ and is unstable when $\mu > \mu_c$. At $\mu = \mu_c$ instability machining operation gives rise to super- and subcritical bifurcations. According to Tobias and his colleagues [2, 3] some mild chatter may be noted generally beyond the left-hand side of the critical depth of cut, in particular, the range $2{\cdot}0 - \mu_c$ in Figure 3b. This is the range of finite-amplitude instability, where the growth or decay of the mild chatter amplitudes are sensitivity dependent upon the initial disturbances.

In this finite-amplitude, range region, the turning operation is dynamically stable for small disturbances and unstable when the disturbances are large. Indeed, the chatter amplitude may shift from one probable value to another. That is, mild chatter starts to build up rapidly in this range until the critical depth of cut μ_c is exceeded. For $\mu > \mu_c$ steady-state turning operation is no longer possible and the corresponding chatter amplitudes will continue to grow exponential until they stabilize themselves at some finite level.

4. Conclusion

The need to eliminate waste and the demand for quality products have motivated a growing interest to understand the nonlinear instability in machining dynamics which are produced as a result of chatter. In this paper, we have not only characterized the bifurcation point, whereby the stability of the linearized turning model is lost through Hopf bifurcation, but also we have identified the super- and subcritical bifurcations leading to finite amplitude instability. In the regions of finite-amplitude, machining is known to be stable for small excitations and unstable when they become large. A notion Tobias and his colleagues [2, 3] have characterized experimentally as a result of the nonlinearity of the cutting force as function of the cutting conditions. Attempts to quantify their findings theoretically, in particular by Hanna and Tobias [2], have been made by Stépán and Nagy [5]. They adopted the aforementioned theory to reduce the infinite character of a nonlinear retarded delay differential equation to ODEs, and using the normal forms they were able to produce similar types of bifurcations for chatter machining. Chatter is of considerable complexity, and the occurring dynamics are far from being clearly characterized. There is indeed wide variety of mechanisms one can consider for the limiting decay or growth of chatter amplitudes, and their transient behaviour, particularly at some finite levels. For instance, anyone of the parametric interactions of the cutting conditions, tool, workpiece and/or machine-tool will impose constraints on satisfactory understanding of the various forms of chatter phenomena and their adverse effects on machining. While a number of investigations have been conducted, and theories and approaches established for characterizing and predicting chatter, there is not a single theory or approach for the study of the stability behaviour of nonlinear chatter excitations for fixed or multiple values of time delays between successive tool cuts. The problem of describing stability behaviour in the generalized centre manifold can be examined for a wide range of the chatter-model parameters. This theory, according to Hale and his colleagues [11–16], is direct, in the sense that, once the ODEs for the DDEs have been obtained in the generalized centre manifold in an infinite-dimensional space, standard techniques for the stability analyses of nonlinear dynamical systems could be employed. In this way, the parameter ranges of the cutting conditions, tool, workpiece and machine-tool over which optimum chip removal processes take place can be determined.

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References

1. S.A. Tobias, *Machine-Tool Vibration*. Blackie: London, Glasgow (1965) 350 pp.
2. N.H. Hanna and S.A. Tobias, A theory of nonlinear regenerative chatter. *ASME J. Eng. Ind.* 96 (1974) 247–255.
3. H.M. Shi and S.A. Tobias, The theory of finite amplitude machine-tool instability. *Proc. Int. J. Machine-Tool Design Res.* 24 (1984) 45–69.
4. J. Tlusty and F. Ismail, Basic nonlinearity in machining chatter. *Annals CIRP* 30 (1981) 1060–1067.
5. G. Stépán and T.K. Nagy, Nonlinear regenerative machine-tool vibrations. *Proc. DETC'97, ASME Design Eng. Tech. Conf.*, September 14–17, Sacramento, California, (1997) 1–11.
6. C.A. van Luttervelt, T.H.C. Childs, I.S. Jawahir, F. Klocke, and P.K. Venuvinod, Present situation and future trends in modelling of machining operations. *Prog. Rep. CIRP Working Group, Modelling of Machining Operations, Annals CIRP* 47 (1998) 587–626.
7. I.E. Minis, E.B. Magrab and I.O. Pandelidis, Improved method for the prediction of chatter in turning, part 1: determination of the structural response parameters. *ASME J. Eng. Industry* 112 (1990) 12–20.

8. N. Saravanja-Fabris and A.F. D'Souza, Nonlinear stability analysis of chatter in metal cutting. *ASME J. Eng. Industry* 96 (1974) 670–675.
9. G.M. Zhang and S.G. Kapoor, Dynamic generation of machined surfaces. Part 1: description of a random excitation system. *ASME J. Eng. Industry* 113 (1991) 137–144.
10. M.M. Nigm, M.M. Sadek and S.A. Tobias, Determination of dynamic cutting coefficients from steady state cutting data. *Proc. Int. J. Machine-tool Design Res.* 17 (1977) 19–37.
11. J.K. Hale and C. Perelló, The neighbourhood of a singular point of functional differential equations. *Contr. Diff. Equations* 3 (1964) 351–375.
12. J.K. Hale, *Theory of Functional Differential Equations*. Berlin: Springer-Verlag (1977) 365 pp.
13. J.K. Hale and S.M.V. Lunel, *Introduction to Functional Differential Equations*. Berlin: Springer-Verlag (1993) 447 pp.
14. J.K. Hale, Critical cases for neutral functional differential equations. *J. Diff. Equations* 10 (1971) 59–82.
15. N. Chafee and J.P. LaSalle, A bifurcation problem for a functional differential equation of finitely retarded type. *J. Math. Anal. Appl.* 35 (1971) 312–348.
16. S.N. Chow and J. Mallet-Paret, Integral averaging and bifurcations. *J. Diff. Equations* 26 (1977) 112–159.
17. J. Bélair and S.A. Campbell, Stability and bifurcations of equilibria in a multiple-delayed differential equation. *SIAM J. Appl. Math.* 54 (1994) 1402–1424.
18. J.R. Baker and K.E. Rouch, Use of finite element structural models in analyzing machine-tool chatter. In: I.S. Jawahir, A.K. Balji and R. Stevenson (eds.), *Proceedings of the CIRP International Workshop on Modelling of Machining Operations*. Atlanta, Georgia (1998) pp. 151–160.
19. N.N. Bogoliubov and Y.A. Mitropolsky, *Asymptotic Methods in Theory of Nonlinear Oscillations*. New York: Gordon and Breach (1960) 535 pp.